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Approximate solution of the Fredholm integral equation of the first kind in a reproducing kernel Hilbert space[☆]

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Abstract

An approach for solving Fredholm integral equations of the first kind is proposed for in a reproducing kernel Hilbert space (RKHS). The interest in this problem is strongly motivated by applications to actual prospecting. In many applications one is puzzled by an ill-posed problem in space $C[a, b]$ or $L^2[a, b]$, namely, measurements of the experimental data can result in unbounded errors of solutions of the equation. In this work, the representation of solutions for Fredholm integral equations of the first kind is obtained if there are solutions and the stability of solutions is discussed in RKHS. At the same time, a conclusion is obtained that approximate solutions are also stable with respect to $\|\cdot\|_\infty$ or $\|\cdot\|_{L^2}$ in RKHS. A numerical experiment shows that the method given in the work is valid.
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1. Introduction

In many fields of science one is interested in continuous functions u which are not directly accessible by experiment, but are related to an experimentally measurable quantity f according to

$$Au = f + \epsilon, \tag{1.1}$$

where A denotes the operator which maps the function u to the experimental data f and ϵ means the measurement errors. In general, problem (1.1) is ill-posed in the sense that the solution of (1.1) does not depend continuously upon the data f, which is often obtained by measurement and hence contains errors.

In many cases A is linear. It may correspond to a Fredholm integral equation of the first kind. Some examples of inverse problems mathematically modeled by a Fredholm integral equation of the first kind are the analysis of

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deep level transient spectroscopy data, the analysis of nuclear magnetic resonance data, the analysis of static light scattering data and so on [1,2]. It is well known that the problem of the stability of the solutions for Fredholm integral equations of the first kind is an ill-posed problem in $C[a,b]$ or $L^2[a,b]$ (for a strict definition of ill-posed problems see Ref. [3]). In order to solve ill-posed problems so-called regularization methods are necessary, i.e. it is essential to find the approximate solution of the ill-posed problems [4,5]. There exist some programs for constructing the regularization operator and choosing an appropriate regularization parameter [6,7]. But these programs are restricted to one specific linear regularization term which models the smoothness constraint.

In this work, we wish to solve the Fredholm integral equation of the first kind in RKHS. The work is organized as follows. A RKHS $W_2^1[a,b]$ is introduced in Section 2. The representation of solutions for Fredholm integral equations of the first kind is obtained in Section 3. In Section 4, stability of solutions of the equation is discussed. A conclusion is obtained that an approximate solution is also stable when perturbations converge to zero with respect to $\|\cdot\|_\infty$ or $\|\cdot\|_{L^2}$ in RKHS. Finally, Section 5 illustrates a numerical experiment related to solving a Fredholm integral equation of the first kind with perturbations.

2. Preliminaries

The space $W_2^1[a,b]$ (see [8]) is defined by

$$W_2^1[a,b] = \{u \mid u : [a,b] \to R,\ u \in AC[a,b],\ u' \in L^2[a,b]\}.$$

The inner product and the norm in $W_2^1[a,b]$ are defined respectively by

$$\langle u(x), v(x)\rangle = \int_a^b u(x)v(x) + u'(x)v'(x)\mathrm{d}x, \quad u, v \in W_2^1[a,b],$$

and

$$\|u\|_{W_2^1[a,b]} = \langle u(x), u(x)\rangle^{1/2}, \quad u \in W_2^1[a,b]. \tag{2.1}$$

$W_2^1[a,b]$ is a complete RKHS and its reproducing kernel is given by

$$R(x,y) = \frac{1}{2\sinh(b-a)}[\cosh(x+y-b-a) + \cosh(|x-y|-b+a)] \tag{2.2}$$

and hence

$$u(\cdot) = \langle u(x), R(x,\cdot)\rangle. \tag{2.3}$$

3. Representation of solutions

In this section, the representation of solutions is given in RKHS for the Fredholm integral equation of the first kind

$$(Au)(x) \triangleq \int_a^b K(x,s)u(s)\mathrm{d}s = f(x), \quad u, f \in W_2^1[a,b] \tag{3.1}$$

where $u(x)$ is unknown function, $f(x)$ is a given function and $K(x,s)$, $\frac{\partial K(x,s)}{\partial x}$ satisfy the conditions

$$\int\int_{[a,b]\times[a,b]} |K(x,s)|^2 \mathrm{d}x\mathrm{d}s \le M_1, \quad M_1 \in R \tag{3.2}$$

and

$$\int\int_{[a,b]\times[a,b]} \left|\frac{\partial K(x,s)}{\partial x}\right|^2 \mathrm{d}x\mathrm{d}s \le M_2, \quad M_2 \in R. \tag{3.3}$$

Lemma 3.1. *The operator A defined in (3.1) is a bounded linear operator from $W_2^1[a,b]$ to $W_2^1[a,b]$ under the conditions (3.2) and (3.3).*

In order to obtain the representation of all the solutions of Eq. (3.1), let $\varphi_i(x) = R(x, x_i)$, where $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[a, b]$. From the definition of the reproducing kernel, we have

$$\langle u(x), \varphi_i(x)\rangle = u(x_i) \tag{3.4}$$

and

$$\begin{aligned}\psi_i(x) &= (A^*\varphi_i)(x) = \langle (A^*R(\cdot, x_i))(s), R(x, s)\rangle \\ &= \langle R(\cdot, x_i), (AR(x, s))(\cdot)\rangle = (AR(x, s))(x_i) \\ &= \int_a^b K(x_i, s)R(x, s)\mathrm{d}s, \quad i = 1, 2, \ldots,\end{aligned}$$

where A^* is the conjugate operator of A. Apply Gram–Schmidt orthonormalization to $\{\psi_i(x)\}_{i=1}^{\infty}$:

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \tag{3.5}$$

where β_{ik} are coefficients of Gram–Schmidt orthonormalization and $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ is an orthonormal system.

Moreover, let $\Psi = \{u | u = \sum_{i=1}^{\infty} c_i\bar{\psi}_i, \text{for}\{c_i\}_{i=1}^{\infty} \in l^2\}$ and P be the projection operator from $W_2^1[a,b]$ to Ψ. Then the solution of (3.1) can be obtained if Eq. (3.1) has a solution in Ψ.

Theorem 3.2. *Let $\{x_i\}_{i=1}^{\infty}$ be dense in the interval $[a, b]$. If Eq. (3.1) has a solution in $W_2^1[a,b]$, then there is a unique solution in Ψ and the solution is expressed as*

$$u_{\min}(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}[\beta_{ik} f(x_k)]\bar{\psi}_i(x). \tag{3.6}$$

Remark 3.3. The symbol $u_{\min}(x)$ denotes the minimal norm solution among all the solutions if the Eq. (3.1) has solutions (this will be proved in Theorem 3.4).

Proof. Let $u(x)$ be a solution of (3.1) in $W_2^1[a,b]$ and $u_{\min}(x) = Pu(x)$. From the definition of P, we have

$$\begin{aligned}u_{\min}(x) &= Pu(x) = \sum_{i=1}^{\infty}\langle u, \bar{\psi}_i\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\left\langle u, \sum_{k=1}^{i}\beta_{ik}A^*\varphi_k\right\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Au, \varphi_k\rangle\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f(x_k)\bar{\psi}_i(x).\end{aligned}$$

Subsequently, we prove that $u_{\min}(x)$ is the solution of (3.1) in Ψ. Since $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[a, b]$, we only prove $(Au_{\min})(x_i) = f(x_i)$. Using the properties of P, $P = P^*$ and $P\psi_i = \psi_i$, one gets

$$\begin{aligned}(Au_{\min})(x_i) = f(x_i) &\Leftrightarrow \langle (Au_{\min})(x), \varphi_i(x)\rangle = f(x_i) \\ &\Leftrightarrow \langle u_{\min}(s), A^*\varphi_i(s)\rangle = f(x_i) \Leftrightarrow \langle (Pu)(s), \psi_i(s)\rangle = f(x_i) \\ &\Leftrightarrow \langle u(s), P(\psi_i(s))\rangle = f(x_i) \Leftrightarrow \langle u(s), \psi_i(s)\rangle = f(x_i) \\ &\Leftrightarrow \langle (Au)(s), \varphi_i(s)\rangle = f(x_i) \Leftrightarrow \langle f(s), \varphi_i(s)\rangle = f(x_i).\end{aligned}$$

The last equation holds from the definition of the reproducing kernel. Therefore, the form (3.6) is the solution of (3.1).

Assume that $u_1(x)$ and $u_2(x)$ are solutions of (3.1) in Ψ. Like in the proof for $u_{\min}(x)$, we have

$$u_1(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f(x_k)\bar{\psi}_i(x). \tag{3.7}$$

and

$$u_2(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f(x_k)\bar{\psi}_i(x). \tag{3.8}$$

Hence $u_1(x) = u_2(x)$. The solution of (3.1) in Ψ is unique. □

Furthermore, we can give the representation of all the solutions for (3.1) if it has solutions.

Let $\Psi^{\perp} = \{u | (u, \Psi) = 0\}$ and define the null space $N(A) = \{u | Au = 0\}$ of A. Then $N(A) = \Psi^{\perp}$. In fact, for any $u(x) \in \Psi^{\perp}$, if

$$0 = \langle u(x), \psi_i(x)\rangle = \langle Au(x), \varphi_i(x)\rangle = Au(x_i), \quad (i = 1, 2, \ldots), \tag{3.9}$$

then we have $Au(x) \equiv 0$ from the density of $\{x_i\}_{i=1}^{\infty}$. Hence $u(x) \in N(A)$. Assume that the sequence $\{r_i\}_{i=1}^{\infty}$ is a basis in $\Psi^{\perp}$. Then the sequence $\{\psi_i\}_{i=1}^{\infty} \bigcup \{r_i\}_{i=1}^{\infty}$ is a complete function system in $W_2^1[a, b]$ and let $\{\bar{\psi}_i\}_{i=1}^{\infty} \bigcup \{\bar{r}_i\}_{i=1}^{\infty}$ be the complete orthonormal function system in $W_2^1[a, b]$ obtained from $\{\psi_i\}_{i=1}^{\infty}$ and $\{r_i\}_{i=1}^{\infty}$. Hence $\{\bar{r}_i\}_{i=1}^{\infty} \perp \Psi$, i.e., $\Psi^{\perp} = \{\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_m, \ldots\}$. Therefore, we have the following theorem.

Theorem 3.4. *If the Eq. (3.1) has solutions, then the representation of all the solutions is given by*

$$u(x) = u_{\min}(x) + \sum_{i=1}^{\infty}\alpha_i \bar{r}_i(x), \tag{3.10}$$

with the real sequence $\{\alpha_i\}_{i=1}^{\infty} \in \ell^2$ and $u_{\min}(x)$ the minimal norm solution of (3.1).

Proof. The form (3.10) is a solution of (3.1) from $\Psi^{\perp} = N(A)$. Here, we only prove that $u_{\min}(x)$ is the minimal norm solution of (3.1). Let u be any solution of (3.1). Then $u = u_{\min} + v$, where $u_{\min} \in \Psi$ and $v \in \Psi^{\perp}$. It holds that $\|u\|^2 = \|u_{\min}\|^2 + \|v\|^2 \geq \|u_{\min}\|^2$. This shows that $u_{\min}(x)$ is the minimal norm solution of (3.1). □

4. Stability

It is well known that the problem of the stability of the solution for Eq. (3.1) is an ill-posed problem in the space $C[a, b]$ or $L^2[a, b]$. In this section, we will discuss it in the reproducing kernel space $W_2^1[a, b]$.

In order to obtain the main theorem, the following lemmas are given.

Lemma 4.1. *The space $\Psi = \{u | u = \sum_{i=1}^{\infty} c_i \bar{\psi}_i,\ for\ \{c_i\}_{i=1}^{\infty} \in \ell^2\}$ is complete in ℓ^2.*

Lemma 4.2. *Let $\overline{A}$ be the restriction of A to Ψ. The inverse operator $\bar{A}^{-1} : W_2^1[a, b] \to \Psi$ exists and is bounded.*

Now, we will give the definition of the stability of the solution for equation (3.1) in $W_2^1[a, b]$.

Definition 4.3. Let $u(x)$ be a solution of (3.1). It is called that the approximate solution $u(x)$ for $u^{(n)}(x)$ with the right-hand side $f^{(n)}(x)$ is stable in $W_2^1[a, b]$ if when $\lim_{n\to\infty} \|f - f^{(n)}\|_{W_2^1[a,b]} = 0$, then $\lim_{n\to\infty} \|u - u^{(n)}\|_{W_2^1[a,b]} = 0$.

From the definition of $\bar{A}$ and Lemma 4.2, the discussion of the stability of any a solution for (3.1) is equivalent to that of the stability of the minimal norm solution.

Theorem 4.4. *If Eq. (3.1) has solutions and let $u_{\min}(x)$ be the minimal norm solution, then the approximate method for obtaining the minimal norm solution $u_{\min}(x)$ from $u_{\min}^{(n)}(x)$ is stable in the reproducing kernel space $W_2^1[a, b]$.*

Proof. If the Eq. (3.1) has solutions, then it has a unique solution in Ψ. Let $\bar{A}u_{\min}^{(n)}(x) = f^{(n)}(x)$ and $f(x) = f^{(n)}(x) + \epsilon^{(n)}(x)$, where $\epsilon^{(n)}(x)$ is a perturbation and $\lim_{n\to\infty} \|\epsilon^{(n)}(x)\|_{W_2^1[a,b]} = 0$.

For $f, f^{(n)} \in W_2^1[a,b]$, from the form (3.6) and

$$u_{\min}^{(n)}(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}[\beta_{ik} f^{(n)}(x_k)]\bar{\psi}_i(x), \tag{4.1}$$

we have

$$u_{\min}(x) - u_{\min}^{(n)}(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}[\beta_{ik}\epsilon^{(n)}(x_k)]\bar{\psi}_i(x). \tag{4.2}$$

On the other hand, since $\bar{A}^{-1}\epsilon^{(n)}(x) \in \Psi$, it follows that

$$\begin{aligned}\bar{A}^{-1}\epsilon^{(n)}(x) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}[\beta_{ik}\langle \bar{A}^{-1}\epsilon^{(n)}(x), \psi_k(x)\rangle]\bar{\psi}_i(x)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}[\beta_{ik}\langle \epsilon^{(n)}(x), (\bar{A}^{-1})^*\psi_k(x)\rangle]\bar{\psi}_i(x).\end{aligned} \tag{4.3}$$

Note that $A^*\varphi_k = \psi_k = P\psi_k \in \Psi$. Hence $\bar{A}^*\varphi_k = \psi_k$ and the right side of above form equals

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{k=1}^{i}[\beta_{ik}\langle \epsilon^{(n)}(x), (\bar{A}^{-1})^*\bar{A}^*\varphi_k(x)\rangle]\bar{\psi}_i(x) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}[\beta_{ik}\langle \epsilon^{(n)}(x), \varphi_k(x)\rangle]\bar{\psi}_i(x)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}[\beta_{ik}\epsilon^{(n)}(x_k)]\bar{\psi}_i(x).\end{aligned} \tag{4.4}$$

Hence

$$u_{\min}(x) - u_{\min}^{(n)}(x) = \bar{A}^{-1}\epsilon^{(n)}(x). \tag{4.5}$$

From the continuity of $\bar{A}^{-1}$ and $\lim_{n\to\infty} \|\epsilon^{(n)}(x)\|_{W_2^1[a,b]} = 0$, we have

$$\lim_{n\to\infty} \|u_{\min}(x) - u_{\min}^{(n)}(x)\|_{W_2^1[a,b]} \le \|\bar{A}^{-1}\|_{W_2^1} \lim_{n\to\infty} \|\epsilon^{(n)}(x)\|_{W_2^1[a,b]} = 0. \quad \square \tag{4.6}$$

The following two important facts follow immediately from the form (4.5).
Define in the set $S = \{u | u \in W_2^1[a,b]\}$

$$\|u\|_{L^2} = \left(\int_a^b u^2(x)\mathrm{d}x\right)^{\frac{1}{2}}. \tag{4.7}$$

Lemma 4.5. *Under the conditions 3.1, $\|\bar{A}^{-1}\|_{L^2}$ is bounded.*

Proof. From the proof of Lemma 3.1, $\|\bar{A}\|_{L^2}$ for the operator $\bar{A} : \Psi \to W_2^1[a,b]$ is bounded under the condition (3.2) and $\|\bar{A}^{-1}\|_{L^2}$ is bounded. $\square$

Corollary 4.6. *In the reproducing kernel space $W_2^1[a,b]$, assume that the conditions of Theorem 4.4 are satisfied. If $\lim_{n\to\infty} \|\epsilon^{(n)}(x)\|_{L^2[a,b]} = 0$, then $\lim_{n\to\infty} \|u_{\min}(x) - u_{\min}^{(n)}(x)\|_{L^2[a,b]} = 0$.*

Proof. Under the conditions of Theorem 4.4, from Lemma 4.5 and the form (4.5), we have

$$\begin{aligned}\|u_{\min}(x) - u_{\min}^{(n)}(x)\|_{L^2} &= \|\bar{A}^{-1}\epsilon_n(x)\|_{L^2}\\ &\le \|\bar{A}^{-1}\|_{L^2}\|\epsilon_n(x)\|_{L^2}.\end{aligned}$$

Hence, $\lim_{n\to\infty} \|u_{\min}(x) - u_{\min}^{(n)}(x)\|_{L^2[a,b]} = 0$ as soon as $\lim_{n\to\infty} \|\epsilon^{(n)}(x)\|_{L^2[a,b]} = 0$. $\square$

Define in $S = \{u | u \in W_2^1[a,b]\}$ again:

$$\|u\|_C = \max_{a\le x\le b} |u(x)|. \tag{4.8}$$

Lemma 4.7. *Under the conditions of Lemma 3.1, if*

$$\max_x \left| \int_a^b K(x,s)\mathrm{d}s \right| \le M, \tag{4.9}$$

then $\|\bar{A}^{-1}\|_C$ is bounded, where M is a constant.

I could write (4.8) as

$$\|u\|_\infty = \sup\{|u(x)| : x \in [a,b]\} \tag{4.10}$$

and then I could write Corollary 4.8 as follows.

Corollary 4.8. *Under the conditions in Theorem 4.4, if $\lim_{n\to\infty} \|\epsilon\|_\infty = 0$, then $\lim_{n\to\infty} \|u_{\min} - u_{\min}^{(n)}\|_\infty = 0$.*

5. Numerical experiments

For Eq. (3.1), it is well known that the Fredholm integral equation of the first kind is an ill-posed problem in $C[a,b]$ or in $L^2[a,b]$. But it is a well-posed problem in $W_2^1[a,b]$ from the above discussion. Here, we give an example as follows:

Example 5.1. Let us consider a 1D heat conduction equation with initial and boundary condition (see [1,9]):

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, t > 0 \\ &u(0,t) = u(\pi,t) = 0, \quad t > 0 \\ &u(x,0) = f(x), \quad 0 \le x \le \pi. \end{aligned} \tag{5.1}$$

The solution of Eq. (5.1) is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-n^2 t} \sin(nx), \qquad a_n = \frac{2}{\pi} \int_0^\pi f(y)\sin(ny)\mathrm{d}y.$$

The inverse heat conduction problem considered involves determining $f(x)$ from the given data $u(\cdot,t)$. Then the problem comes down to solving the Fredholm integral equation of the first kind

$$\int_0^\pi k(x,y) f(y)\mathrm{d}y = u(x,t), \quad 0 \le x \le \pi, \tag{5.2}$$

where $k(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t}\sin(nx)\sin(ny)$. The problem on solving equation (5.2) is an ill-posed problem in the space $L^2[0,\pi]$. Here, we solve the approximate solution of Eq. (5.2) with the right-hand side given a perturbation in $W_2^1[a,b]$.

Take $t = 1$, $u(x,1) = \sum_{n=1}^{\infty} \frac{2\sin(n\pi)}{\pi - n^2\pi} \mathrm{e}^{-n^2} \sin(nx)$; the true solution is $f(x) = \sin x$. We calculate the approximate solution $\hat{f}(x)$. All computations are performed using the Mathematica software package. If nodes are chosen as $x = (2i-1)\pi/12$, $i = 1, 2, \ldots, 6$, then we present the maximum absolute error $\max|f(x) - \hat{f}(x)|$ in Table 5.1 when the right-hand side of Eq. (5.2) is put on perturbations $\epsilon = 0.05$ and $\epsilon = 0.005$, separately, in the space $W_2^1[0,\pi]$.

From the above numerical examples, we can see that the solution is stable when the right-hand side is given small perturbations. This illustrates that the new method given in the work is valid.

Table 5.1
Computational results of Example 5.1

ϵ	$\epsilon = 0.05$	$\epsilon = 0.005$
$\max \lvert f(x) - \hat{f}(x) \rvert$	4.28E−02	4.112E−03

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